Scientific Data Compression with Adaptive hp-FEM

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`http://hpfem.org/himg/`

PDE, FEM, and hp-FEM

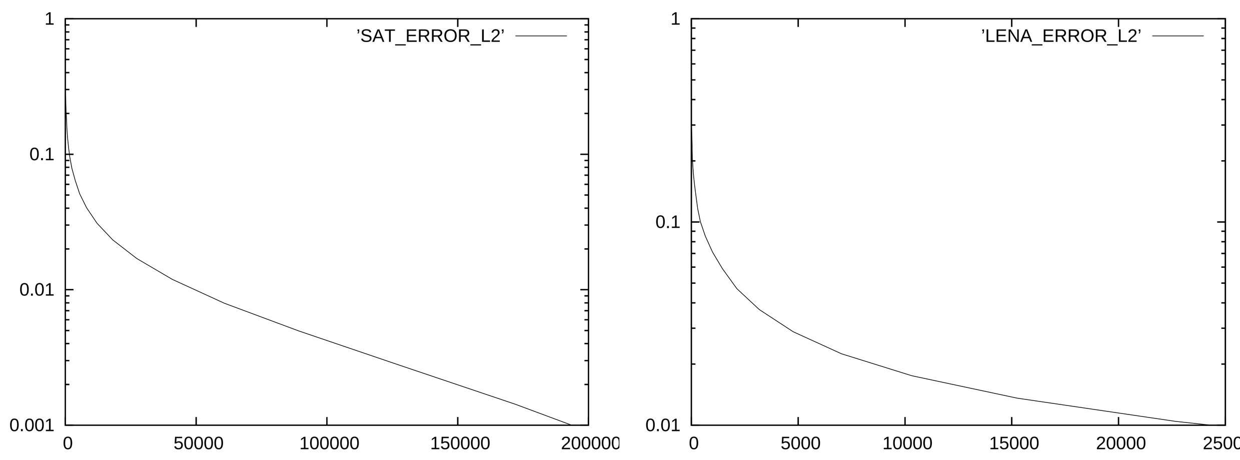
Many important natural processes, such as the weather, flow of liquids, deformation of solid bodies, heat transfer, effects of electromagnetic fields, and others, are described by *partial differential equations (PDE)*. With few exceptions, PDE cannot be solved exactly, and therefore they have to be solved approximately by means of sophisticated computational methods. The *Finite Element Method (FEM)* is the most powerful numerical method for the approximate solution of PDE. It is based on partitioning the computational domain into small and geometrically simple objects, *finite elements*, where the solution to the PDE (such as fluid velocity, density, pressure, displacement, stress, temperature, electromagnetic field, etc.) is approximated by polynomials. The piecewise-polynomial approximation is characterized by a finite number of unknown real coefficients. These constants are called *degrees of freedom (DOF)*. In the hp-FEM (also called hp-version of the FEM), both the size h and polynomial degree p of elements moreover are adapted automatically in order to maximize the convergence speed. By maximizing convergence speed we mean to reduce the approximation error as fast as possible with respect to the number of DOF used. The hp-FEM can attain *exponential rate of convergence* $ERR \approx e^{-DOF}$ which is the fastest known convergence rate in FEM analysis.

Application to Image Compression

The finite element method can be applied to image compression naturally. The computational domain Ω is a rectangle or hexahedron containing the image. A greyscale image is a piecewise-constant function defined in Ω, with values between 0 and 255 in pixels. A color image consists of three such functions for the red, green, and blue components. The image plays the role of an *exact solution* to a PDE. Instead of solving a PDE that describes a physical process, we solve a PDE that performs an *orthogonal projection*, i.e., finds the best approximation of the image in the finite element space. The algorithm is capable of compressing both pixel-wise constant and continuous data. The adaptive algorithm starts with an extremely coarse mesh, typically just one element. One step of the algorithm looks as follows: (1) determine elements with largest projection error (2) determine optimal refinement of every such element (3) refine the elements in question (4) project the image on the new mesh.

Error Control and Convergence

The algorithm makes is possible to measure the approximation error (data loss) in a mathematically sound way. The compression can be either lossy or lossless.



Convergence of the adaptive process for the satellite photo and Lena.

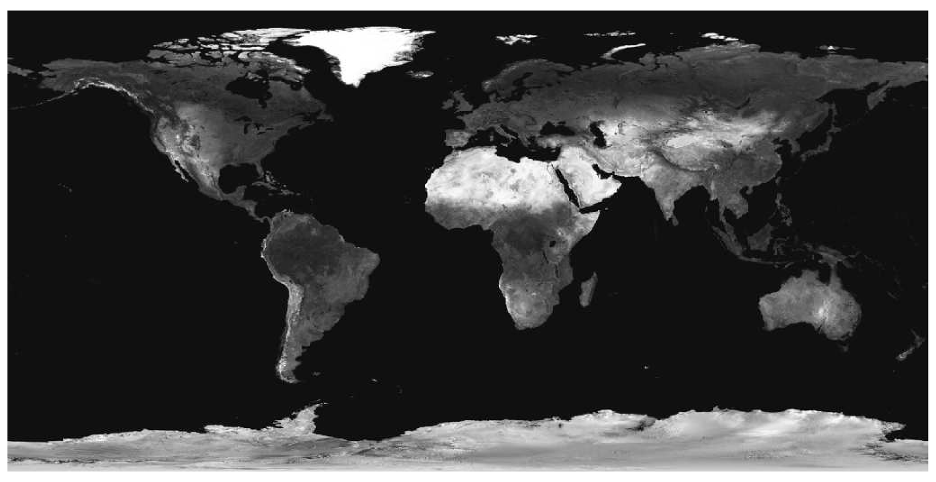
Current State and Outlook

The FEM algorithm does not contain any heuristics that are typical for traditional image compression algorithms. It can compress more general data than piecewise-constant functions, and it controls the error of the compression. The compressed images are stored as continuous, piecewise-polynomial functions, and thus many existing visualization methods and software can be applied *directly to the compressed images*.

On the other hand, this study is just a proof of concept, and the algorithm is slower than traditional image compression algorithms. **There is much room for improvement, visit the project home page** `http://hermes.org/himg` **and GET INVOLVED!**

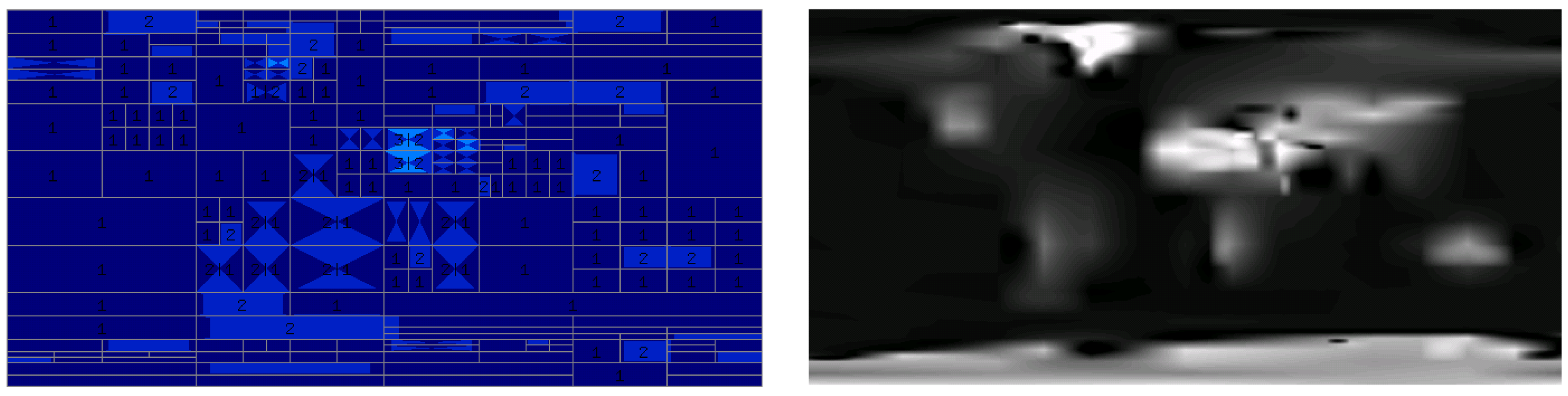
Example: Compression of Satellite Image of Earth

Performance of the adaptive algorithm can be illustrated on a high-resolution satellite image of Earth (courtesy NASA).

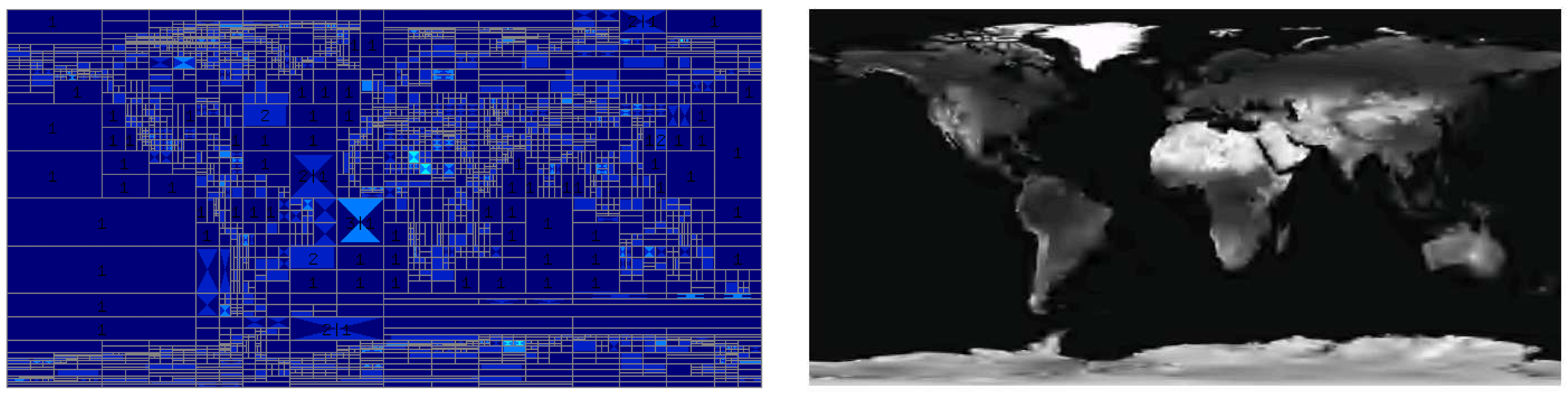


Satellite photograph of Earth, size 1024×512 pixels.

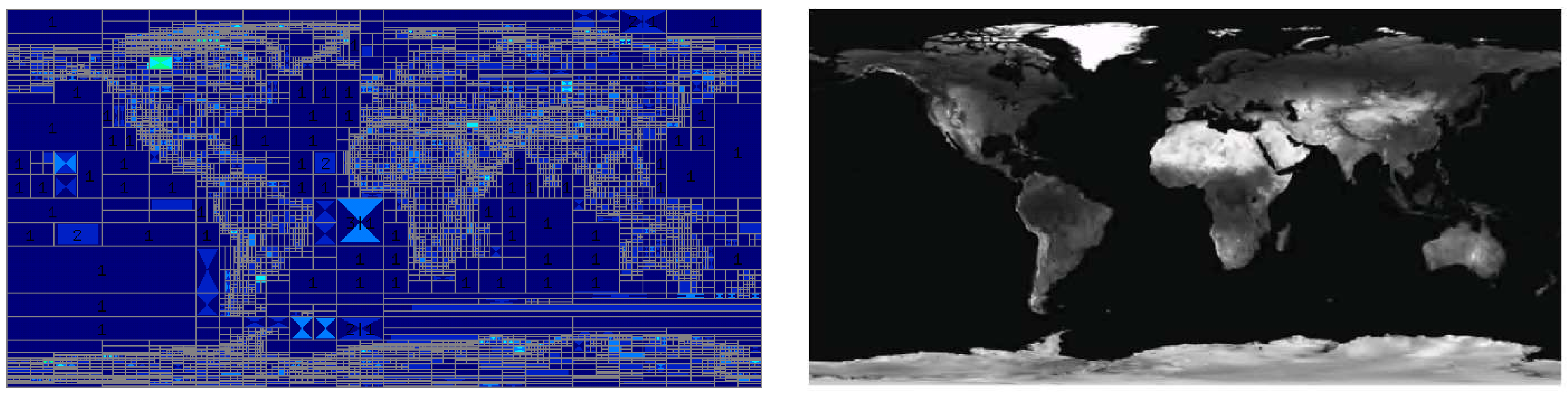
Below we show approximations and finite element meshes corresponding to relative error ERR of 18%, 6%, 3%, and 1% in the L^2-norm. The symbol "DOF" stands for *degrees of freedom* (parameters defining the approximation). Notice that the algorithm leaves large elements in areas where "nothing happens" while small elements only are used to resolve small-scale phenomena. In contrast to that, JPEG always subdivides the image into small cella of 8×8 pixels.



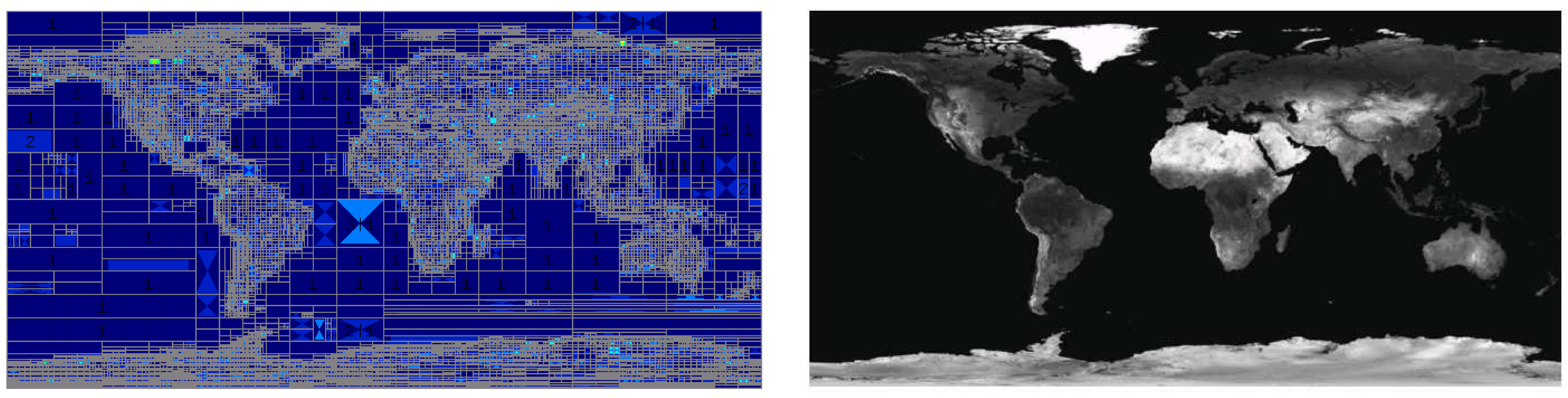
Finite element mesh and approximation, $DOF = 419$, $ERR = 18.2095\%$.



Finite element mesh and approximation, $DOF = 3778$, $ERR = 6.3895\%$.



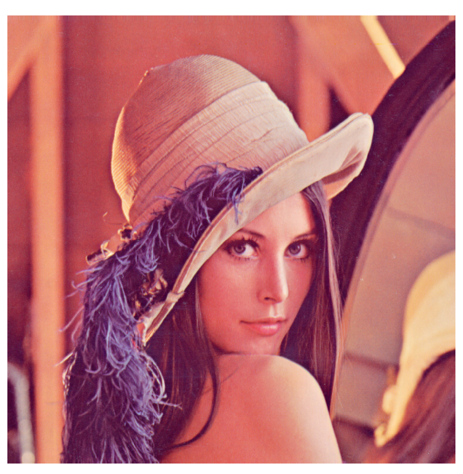
Finite element mesh and approximation, $DOF = 12125$, $ERR = 3.0938\%$.



Finite element mesh and approximation, $DOF = 40901$, $ERR = 1.1916\%$.

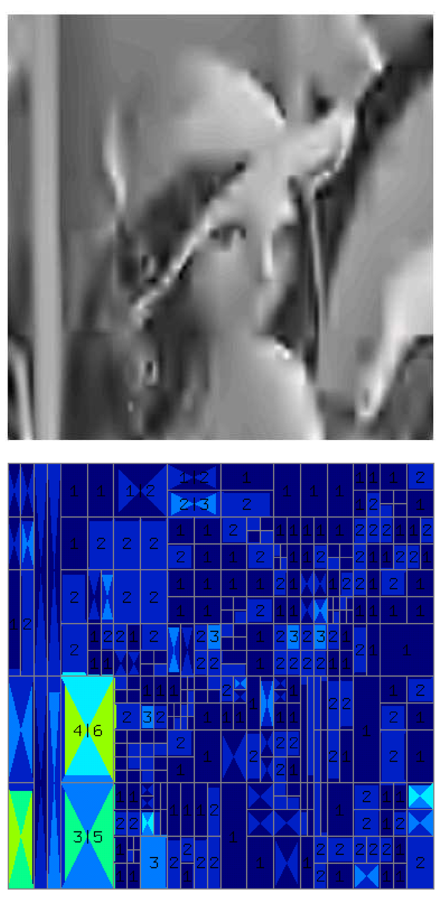
Example: Lena

This is a traditional benchmark problem in the image compression community, (upper part of a) photo of November 1972 Playmate Lena Sjööblom.



Lena, size 512×512 pixels.

This image contains multiple phenomena that cause problems to image compression algorithms.



$DOF = 420$,
$ERR = 10.0344\%$.

$DOF = 2129$,
$ERR = 4.7012\%$.

$DOF = 22372$,
$ERR = 1.0449\%$.

References

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[2] P. Solin, K. Segeth, I. Dolezel: *Higher-Order Finite Element Methods*, 408 pages, Chapman & Hall/CRC Press, July 2003, ISBN 158488438X.

[3] P. Solin, J. Cerveny, I. Dolezel: Arbitrary-Level Hanging Nodes and Automatic Adaptivity in the hp-FEM, Math. Comput. Simul. 77 (2008), 117 - 132.

[4] P. Solin, D. Andrs, J. Cerveny, M. Simko: PDE-Independent Adaptive hp-FEM Based on Hierarchic Extension of Finite Element Spaces, J. Comput. Appl. Math. 233 (2010) 3086-3094.