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On the stability analysis of nonlinear systems using polynomial Lyapunov functions

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Abstract

In the stability study of nonlinear systems, not to found feasible solution for the LMI problem associated with a quadratic Lyapunov function shows that it doesn't exist positive definite quadratic Lyapunov function that proves stability of the system, but doesn't show that the system isn't stable. So, we must search for other Lyapunov functions. That's why, in the present paper, the construction of polynomial Lyapunov candidate functions is investigated and sufficient conditions for global asymptotic stability of analytical nonlinear systems are proposed to allow computational implementation.

The main keys of this work are the description of the nonlinear studied systems by polynomial state equations, the use of an efficient mathematical tool: the Kronecker product; and the non-redundant state formulation. These notations allow algebraic manipulations and make easy the extension of the stability analysis associated to quadratic or homogeneous Laypunov functions towards more general functions.

The advantage of the proposed approach is that the derived conditions proving the stability of the studied systems can be presented as linear matrix inequalities (LMIs) feasibility tests and the obtained results can show in some cases how the polynomial Lyapunov functions leads to less conservative results than those obtained via quadratic (QLFs) or monomial Laypunov functions. This contribution to the stability analysis of high order nonlinear continuous systems can be extended to the stability, robust analysis and control of other classes of systems.

Keywords: Nonlinear system; Kronecker product; Stability; Polynomial lyapunov function; LMIs

1. Introduction

The problem of stability analysis of the nonlinear systems has received considerable attention and has been largely reported in the literature [1,4,6,7,10,13–17,21]. A lot of them focus on analysis approaches in the framework of linear matrix inequalities (LMIs) [11,13,14,16]. The design approaches range from using quadratic Lyapunov function [1,5,11,16,17] to those based on monomial or polynomial Lyapunov functions [13,18]. However the proposed results remain restrictive to the linear systems and particular classes of nonlinear models, and there is no method for studying general high order nonlinear systems. The main problem is to have a useful notation which allows algebraic manipulations in a general form.

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For the stability analysis of the nonlinear studied systems, we use the Lyapunov's direct method and tend to formulate the stability conditions in terms of a set of linear matrix inequalities. However, if no feasible solution is found for the LMI problem associated with a quadratic Lyapunov function, this shows that there exists no positive definite quadratic Lyapunov function proving stability of the system but do not show that it is not stable. So, we must look for other Lyapunov functions.

Our motivation for this problem can be illustrated by the following example:

Example [19] Consider the nonlinear system with polynomial vector fields described by the following equations:

$$\begin{cases} \dot{x}_1 = -x_1^3 + 4x_2^3 - 6x_3x_4 \\ \dot{x}_2 = -x_1 - x_2 + x_5^3 \\ \dot{x}_3 = x_1x_4 - x_3 + x_4x_6 \\ \dot{x}_4 = x_1x_3 + x_3x_6 - x_4^3 \\ \dot{x}_5 = -2x_2^3 - x_5 + x_6 \\ \dot{x}_6 = -3x_3x_4 - x_5^3 - x_6 \end{cases}$$

which has an equilibrium at the origin. As a first attempt, the construction of a quadratic Lyapunov function had been tried but no function of this form is found. Failure to find a quadratic Lyapunov function does not necessarily mean that the equilibrium is unstable, as the conditions are sufficient. Indeed, the following function $V(X)$:

$$V = 0.7257x_1^2 + 1.3x_2^4 + 2.325x_3^2 + 1.575x_4^2 + 0.65x_5^4 + 1.3x_6^2$$

is given as a polynomial Lyapunov function of the above system.

We have seen from this example that the problem of stability analysis can be solved if we can determine a polynomial Lyapunov function. We thus pose the following question: under what conditions can such a polynomial Lyapunov function be found.

So, in the present work, the construction of polynomial Lyapunov candidate functions is investigated and a corresponding approach is described. We rely on the description of the nonlinear analytical systems by polynomial state equations [3,8,9,15] and the use of the Kronecker product mathematical tool [12], its useful rules and the non-redundant state formulation [20].

The present paper is organized as follows: Section 2 contains the mathematical notations and preliminary material on algebraic properties of the Kronecker product. The description of the nonlinear studied systems and problem formulation are presented in Section 3 where relationships with previous work are also discussed. Some preliminary results are derived in Section 4. A theorem and sufficient conditions for global asymptotic stability of analytical nonlinear systems as LMI problems are proposed in Section 5. The derived conditions are established using the Lyapunov's direct method associated in a first time with homogeneous function and with a general polynomial Lyapunov function in a second time. Some concluding remarks are given in Section 6.

2. Mathematical notations and preliminaries

In this section, we present some needed rules and functions and establish the mathematical notations for later use. The dimensions of the matrices used here are the following:

$$A(p\times q),\ B(r\times s),\ C(q\times g),\ D(s\times h),\ E(n\times p),\ P(n\times n),\ X(n\times 1)\in\Re^n,\ Y(m\times 1)\in\Re^m,\ Z(q\times 1)\in\Re^q.$$

Let's consider the following notations: I_n: $(n\times n)$ identity matrix; $0_{n\times m}$: $(n\times m)$ zero matrix; 0: zero matrix of convenient dimensions; A^{T}: transpose of matrix A; $A>0$ $(A\geq 0)$: symetric positive definite (semi-definite matrix A); $e_{\underset{(q)}{k}}$: q dimensional unit vector which has 1 in the kth element and zero elsewhere;

- The kth row of a matrix such as A is denoted $A_{\cdot k}$ and the kth column is denoted $A_{k\cdot}$. The ik element of A will be denoted a_{ik}
- The Kronecker product of A and B is denoted $A\otimes B$ a $(p{\cdot}r\times q{\cdot}s)$ matrix, and the i-th Kronecker's power of A denoted $A^{[i]}=A\otimes A\otimes\ldots\otimes A$ is a $(p^i\times q^i)$ matrix

- The non-redundant$j$-power $\tilde{X}^{[j]}$ of the state vector X introduced in [20] is defined as:

$$\left\{ \overset{\tilde{X}^{[1]}=X^{[1]}=X}{\forall j \geq 2 \tilde{X}^{[j]} = [x_1^j, x_1^{j-1}x_2, \ldots, x_1^{j-1}x_n, x_1^{j-2}x_2^2 x_1^{j-2}x_2x_3, \ldots, x_1^{j-2}\, x_2x_n, \ldots x_1^{j-2}x_n^2, \ldots, x_1^{j-3}x_2^3, \ldots x_n^j]^T} \right.$$

where the repeated components of the redundant j-power $X^{[j]}$ are omitted. Then we have the following relation:

$$\begin{cases} \forall j \in \aleph\, \exists! T_j \in \Re^{n^j \times \alpha_j}; \alpha_j = \begin{pmatrix} n+j-1 \\ j \end{pmatrix} \\ X^{[j]} = T_j \tilde{X}^{[j]} \end{cases} \tag{1}$$

thus, one possible solution for the inversion can be written as:

$$\tilde{X}^{[j]} = T_j^+ X^{[j]} \tag{2}$$

where T_j^+ is the Moore-Penrose pseudo inverse of T_j given by

$$T_j^+ = (T_j^T T_j)^{-1} T_j^T \tag{3}$$

and α_j stands for the binomial coefficients. A procedure of the determination of the matrix T_j has been proposed in [2].

- The permutation matrix denoted $U_{n\times m}$ is defined in [12]

$$U_{n\times m} = \sum_{i=1}^{n}\sum_{k=1}^{m} (\underset{(n)}{e_i}\, \underset{(m)}{e_k^T}) \otimes (\underset{(m)}{e_k}\, \underset{(n)}{e_i^T}) \tag{4}$$

  This matrix is square ($nm \times nm$) and has precisely a single 1 in each row and in each column.
- An important vector valued function of matrix denoted $vec(\cdot)$ was defined as [12]:

$$\underset{(pq\times 1)}{vec(A)} = \begin{bmatrix} A_{\cdot 1} \\ A_{\cdot 2} \\ \vdots \\ A_{\cdot q} \end{bmatrix} \tag{5}$$

- A matrix valued function is a vector denoted $\text{mat}_{(n,m)}\,(\cdot)$ was defined in [1] as follows:

  if V is a vector of dimension $p = n \times m$ then $M = \text{mat}_{(n,m)}(V)$ is the $(n \times m)$ matrix verifying : $V = vec(M)$ (6)

- Among the main properties of this product presented in [12], we recall the following useful ones:

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD) \tag{7}$$

$$(A \otimes B)^T = A^T \otimes B^T \tag{8}$$

$$B \otimes A = U_{r\times p}(A \otimes B)U_{q\times s} \tag{9}$$

$$X \otimes Y = U_{n\times m}(Y \otimes X) \tag{10}$$

$$vec(EAC) = (C^T \otimes E)vec(A) \tag{11}$$

$$vec(A^T) = U_{p\times q}vec(A) \tag{12}$$

$$\forall\, i \leq k\; X^{[k]} = U_{n^i \times n^{k-i}} X^{[k]} \tag{13}$$

3. Description of the studied systems and problem formulation

In this work, we focus on analytical nonlinear continuous systems described by the following state-space equation:

$$\dot{X}(t) = F(X(t)) \quad (14)$$

where $\forall t \in \Re^{+}$; $X(t) \in \Re^{n}$ is the state vector and $F(\cdot)$ is an analytic vector field from $\Re^{n}$ into $\Re^{n}$. We suppose that $X = 0$ is the unique equilibrium of the system (14).

With the Kronecker power of vectors, the analytic vector $F(\cdot)$ can be written as:

$$F(X) = \sum_{i=1}^{\infty} F_i X^{[i]} \quad (15)$$

and F_i are constant matrices of adapted dimensions.

In practice, we consider a truncated r-form [1,9,20] of the development (15) and then we study the nonlinear system defined by the following polynomial state space equation:

$$\dot{X} = \sum_{i=1}^{r} F_i X^{[i]} \quad (16)$$

The basic problem addressed in this paper is the construction of a Lyapunov function proving global asymptotic stability of the nonlinear system (16).

A large amount of work has been developed in the stability area considering a particular class of nonlinear systems where the Lyapunov functions are chosen to be quadratic or to be a monomial function in some recent works concerned with the study of uncertain systems [14,21].

The present paper is an attempt towards expanding the class of candidate functions to general polynomial ones proving the stability of the analytical nonlinear continuous studied systems that can failed the feasibility problem when using quadratic Lyapunov functions ($V(X) = X^T PX$).

More specifically, our aim is to find a polynomial Lyapunov function such that:

(i) $V(X) > 0$
(ii) $\dot{V}(X) < 0$

and to establish sufficient conditions for the stability of the studied systems (16) as *LMI* problems.

4. Preliminary results

Throughout this work, we will exploit the derived following lemmas needed for the stability analysis, of the studied systems, in the next sections.

Lemma 1. *Consider the nonlinear system described by:*

$$\dot{X} = \sum_{i=1}^{r} F_i X^{[i]}$$

where $F_i \in \Re^{n \times n^i}$, $i = 1, \ldots, r$, *are constant matrices. Then* $F_{i[\mathrm{p}]}$ $i = 1, \ldots, r$ are ($n^p \times {}^{p+i-1}$) *matrices in the equation:*

$$\frac{\mathrm{d}}{\mathrm{d}t}(X^{[p]}) = \sum_{i=1}^{r} F_{i[p]} X^{[i+p-1]} \quad (17)$$

and $\tilde{F}_{i[p]} i = 1, \ldots, r$ are ($\alpha_p \times \alpha_{p+i-1}$) *matrices in the equation:*

$$\frac{\mathrm{d}}{\mathrm{d}t}(\tilde{X}^{[p]}) = \sum_{i=1}^{r} \tilde{F}_{i[p]} \tilde{X}^{[i+p-1]} \quad (18)$$

verifying the following relations:

$$\begin{cases} F_{i[p]} = & (F_{i[p-1]} \otimes I_n) + U_{n^{p-1} \times n}(F_i \otimes I_{n^{p-1}}) \\ & \tilde{F}_{i[p]} = T_p^+[F_{i[p]}]T_{p+i-1} \\ & F_{i[1]} = \tilde{F}_{i[1]} = F_i \end{cases} \tag{19}$$

with: T_p, T_p^+ and α_p *defined, respectively, in* (1)–(3)

Proof. see Appendix A.1 □

Lemma 2. *Let* $X(n \times 1)$, $Y(m \times 1)$ *two vectors. If* $Z(q \times 1)$ *is a vector of dimension* $q = n.m$, *then we have the following relation:*

$$Y^T \underset{(m,n)}{\mathrm{mat}}(Z)X = Z^T(X \otimes Y) \tag{20}$$

Proof. *the proof of this lemma is given in the* Appendix A.2. □

Lemma 3. $A \in \Re^{(p\times q)}$, $Z \in \Re^{(q\times 1)}$; $p = n.m$

$$\underset{(n,m)}{\mathrm{mat}}(AZ) = (Z^T \otimes I_n) \begin{bmatrix} \underset{(n,m)}{\mathrm{mat}}(A_{.1}) \\ \vdots \\ \underset{(n,m)}{\mathrm{mat}}(A_{.q}) \end{bmatrix} \tag{21}$$

Proof. refer to Appendix A.3 □

5. The proposed sufficient conditions for global asymptotic stability of nonlinear systems-LMI problems

5.1. Homogeneous candidate Lyapunov functions

Let us consider the following homogeneous Lyapunov function:

$$V(X) = X^{[p]^T} P X^{[p]} \tag{22}$$

which is positive definite when P is a symmetric positive definite $(n^p \times n^p)$matrix [16].

Differentiating along trajectory of the system (16), we have:

$$\dot{V}(X) = X^{[p]^T} P \frac{\mathrm{d}}{\mathrm{d}t}(X^{[p]}) + \frac{\mathrm{d}}{\mathrm{d}t}(X^{[p]^T})PX^{[p]} = 2\sum_{k=1}^{r} X^{[p]^T} PF_{k[p]}X^{[p+k-1]} \tag{23}$$

Using the rule of the *vec*-function (11), the relation (23) can be written as:

$$\dot{V}(X) = 2\sum_{k=1}^{r} V_k^T X^{[2p+k-1]} \tag{24}$$

where

$$V_k = vec(PF_{k[p]}) \tag{25}$$

Knowing that all polynomials with even degree can be represented as a symmetric quadratic form [13,21]. Thus, we assume in the following development that r is odd: $r = 2s + 1$, and we can write:

$$V_k^T X^{[2p+k-1]} = \sum_{j=g_k}^{h_k} \lambda_{p+k-j-1,p+j} X^{[p+k-j-1]^T} N_{p+k-j-1,p+j} X^{[p+j]} \tag{26}$$

where $\lambda_{p+k-j-1,p+j}$ are arbitrary reals verifying:

$$\sum_{j=g_k}^{h_k} \lambda_{p+k-j-1,p+j} = 1 \tag{27}$$

and

$$\text{for} \quad k = 1, \ldots, 2s+1 : \; g_k = \sup(0, k-s-1) \quad \text{and} \quad h_k = \inf(s, k-1) \tag{28}$$

$$\text{for} \quad j = g_k, \ldots, h_k : N_{p+k-j-1,p+j} = \underset{(n^{p+k-j-1}, n^{p+j})}{\text{mat}} (V_k) \tag{29}$$

Applying the properties [1,2], one obtains:

$$N_{p+k-j-1,p+j} = \underset{(n^{p+k-j-1}, n^{p+j})}{\text{mat}} (vec(PF_{k[p]})) = U_{n^{k-j-1} \times n^p} (P \otimes I_{n^{k-j-1}}) M_{p+k-j-1,p+j} \tag{30}$$

with

$$M_{p+k-j-1,p+j} = \begin{bmatrix} \underset{(n^{k-j-1}), n^{p+j}}{\text{mat}} (F^T_{k[p]1.}) \\ \vdots \\ \underset{(n^{k-j-1}), n^{p+j}}{\text{mat}} (F^T_{k[p]n^p.}) \end{bmatrix} \tag{31}$$

The result (31) and the relation (30) allow us to write:

$$\begin{aligned} X^{[p+k-j-1]^T} N_{p+k-j-1,p+j} \, X^{[p+j]} &= X^{[p+k-j-1]^T} U_{n^{k-j-1} \times n^p} (P \otimes I_{n^{k-j-1}}) M_{p+k-j-1,p+j} X^{[p+j]} \\ &= X^{[p+k-j-1]^T} (P \otimes I_{n^{k-j-1}}) M_{p+k-j-1,p+j} X^{[p+j]} \end{aligned} \tag{32}$$

Consequently, we have:

$$V_k^T X^{[2p+k-1]} = \sum_{j=g_k}^{h_k} \lambda_{p+k-j-1,p+j} \, X^{[p+k-j-1]^T} N_{p+k-j-1,p+j} \, X^{[p+j]} = \mathbf{X}^T (\mathbf{P}\mathbf{M}_k) \mathbf{X} \tag{33}$$

with

$$\mathbf{X} = \begin{bmatrix} X^{[p]} \\ X^{[p+1]} \\ \vdots \\ X^{[p+s]} \end{bmatrix} \quad \text{and} \quad \mathbf{P} = \begin{bmatrix} P & & & 0 \\ & P \otimes I_n & & \\ & & \ddots & \\ 0 & & & P \otimes I_{n^s} \end{bmatrix} \tag{34}$$

Let's note that $\mathbf{P}$ is a symmetric positive matrix, and $\dot{V}(X)$ (24) can be written as:

$$\dot{V}(X) = 2 \sum_{k=1}^{2s+1} V_k^T \, X^{[2p+k-1]} = \mathbf{X}^T (\mathbf{P}\mathbf{M} + \mathbf{M}^T \mathbf{P}) \mathbf{X} \tag{35}$$

with

$$\mathbf{M} = \sum_{k=1}^{2s+1} \mathbf{M}_k = \begin{bmatrix} \lambda_{p,p} M_{p,p} & \lambda_{p,p+1} M_{p,p+1} & \cdots & \lambda_{p,s+p} M_{p,s+p} \\ \vdots & \ddots & & \\ \vdots & & \ddots & \\ \lambda_{s+p,p} M_{s+p,p} & & & \lambda_{s+p,s+p} M_{s+p,s+p} \end{bmatrix} \tag{36}$$

When considering the non-redundantform, the vector $\mathbf{X}$ can be defined by:

$$\mathbf{X} = \tau\tilde{X} \tag{37}$$

where

$$\tau = \begin{bmatrix} T_p & & 0 \\ & \ddots & \\ 0 & & T_{p+s} \end{bmatrix} \quad \text{and} \quad \tilde{X} = \begin{bmatrix} \tilde{X}^{[p]} \\ \vdots \\ \tilde{X}^{[p+s]} \end{bmatrix} \tag{38}$$

The final expression of $\dot{V}(X)$ is as follows:

$$\dot{V}(X) = \mathbf{X}^T\tau^T(\mathbf{PM} + \mathbf{M}^T\mathbf{P})\tau\mathbf{X} \tag{39}$$

A sufficient condition of the global asymptotic stability of the equilibrium $X = 0$ is that $\dot{V}(X)$ (39), be negative definite. Considering the obtained results, we can derive *LMI*'s sufficient conditions for global asymptotic stability of the studied systems:

Lemma 4. *The system in* (16) *is stable, if there exits a feasible solution to the LMIs*

$$\exists\mathbf{P} = \mathbf{P}^T \quad \mathbf{P} > 0 \quad \mathbf{PM} + \mathbf{M}^T\mathbf{P} < 0 \tag{40}$$

τ, $\mathbf{M}$ and $\mathbf{P}$ *are defined in* (34), (36) and (38).

Moreover, the Lyapunov function proving the stability is given by: $V(X) = X^{[p]^T}PX^{[p]}$

If no positive definite homogenous Lyapunov function is found, we have to look in the following sub-section for the construction of more general polynomial Lyapunov candidate functions to investigate the stability of the studied systems (16).

5.2. General polynomial Lyapunov functions

Let us consider the following Lyapunov function and a $(n_0 \times n_0)$ symmetric positive definite matrix $\hat{P}$ such that:

$$V(X) = \hat{X}^T\hat{\mathbf{P}}\hat{X} \text{ with } \hat{X} = \begin{bmatrix} \tilde{X} \\ \tilde{X}^{[2]} \\ \vdots \\ \tilde{X}^{[p]} \end{bmatrix} \tag{41}$$

Let consider the following notations used in this section:

$$\begin{cases} n_0 = \sum_{i=1}^{p}\alpha_i,\ n_k \underset{(k=1,\ldots,p)}{=} \sum_{i=k}^{k+r-1}\alpha_i \\ m_k \underset{(k=1,\ldots,p+r-1)}{=} \sum_{i=1}^{p}\alpha_{i+k},\ m_{s_k} \underset{(k=1,\ldots,p+s)}{=} \sum_{i=1}^{p+s}\alpha_{i+k},\ n_s = \sum_{i=1}^{p+s}\alpha_i\, n_{\Sigma_1} = \sum_{k=1}^{p}n_k,\ n_{\Sigma_2} = \sum_{k=1}^{p+r-1}m_k,\ n_{\Sigma_3} = \sum_{k=1}^{p+s}m_{s_k}, \\ N_2 = \sum_{i=2}^{2p+r-1}\alpha_i \end{cases}$$

The derivative of $V(X)$ (41) is:

$$\dot{V}(X) = \hat{X}^T\hat{P}\dot{\hat{X}} + \dot{\hat{X}}^T\hat{P}\hat{X} \tag{43}$$

It can be easily checked that $\dot{\hat{X}}$ can be written as (refer to Appendix A.4):

$$\dot{\hat{X}} = A.H \begin{bmatrix} \tilde{X} \\ \tilde{X}^{[2]} \\ \vdots \\ \tilde{X}^{[p+r-1]} \end{bmatrix} \tag{44}$$

where

$$A = diag\left[A_1 \mid \cdots \mid A_k \quad \cdots \quad A_p \right]$$
$$\text{with } A_k = \left[\tilde{F}_{1[k]} \cdots \tilde{F}_{r[k]} \right] \tag{45}$$

and

$$H = \begin{bmatrix} I_{n_1} & & 0_{n_1\times(n_p-n_1)} \\ 0_{n_2\times\alpha_1} & I_{n_2} & 0_{n_2\times(n_p-n_2-\alpha_1)} \\ 0_{n_3\times(\alpha_1+\alpha_2)} & I_{n_3} & 0_{n_3\times(n_p-\alpha_1-\alpha_2-n_3)} \\ \vdots & \vdots & \vdots \\ 0_{n_p\times(\alpha_1+\alpha_2+....+\alpha_{p-1})} & & I_{n_p} \end{bmatrix} \tag{46}$$

Therefore, the equality (43) yields the following ones:

$$\dot{V}(X) = \begin{bmatrix} \tilde{X} \\ \tilde{X}^{[2]} \\ \vdots \\ \tilde{X}^{[p]} \end{bmatrix}^T \hat{P}AH \begin{bmatrix} \tilde{X} \\ \tilde{X}^{[2]} \\ \vdots \\ \tilde{X}^{[p+r-1]} \end{bmatrix} + \begin{bmatrix} \tilde{X} \\ \tilde{X}^{[2]} \\ \vdots \\ \tilde{X}^{[p+r-1]} \end{bmatrix}^T H^T A^T \hat{P} \begin{bmatrix} \tilde{X} \\ \tilde{X}^{[2]} \\ \vdots \\ \tilde{X}^{[p]} \end{bmatrix} \tag{47}$$

Using the *vec*-function, the derivative of the Lyapunov function (47) can be written as:

$$\dot{V}(X) = 2\,\mathrm{vec}^T(\hat{P}AH) \begin{bmatrix} \tilde{X} \\ \tilde{X}^{[2]} \\ \vdots \\ \tilde{X}^{[p+r-1]} \end{bmatrix} \otimes \begin{bmatrix} \tilde{X} \\ \tilde{X}^{[2]} \\ \vdots \\ \tilde{X}^{[p]} \end{bmatrix} \tag{48}$$

To remove the redundant elements in the previous expression, we introduce the following equation which is obvious to check (see Appendix A.5):

$$\begin{bmatrix} \tilde{X} \\ \tilde{X}^{[2]} \\ \vdots \\ \tilde{X}^{[p+r-1]} \end{bmatrix} \otimes \begin{bmatrix} \tilde{X} \\ \tilde{X}^{[2]} \\ \vdots \\ \tilde{X}^{[p]} \end{bmatrix} = \bar{U} \begin{bmatrix} \tilde{X}^{[2]} \\ \tilde{X}^{[3]} \\ \vdots \\ \tilde{X}^{[2p+r-1]} \end{bmatrix} \tag{49}$$

where

$$\bar{U} = \begin{bmatrix} U_{\alpha_1\times n_0} & & & 0 \\ & U_{\alpha_2\times n_0} & & \\ & & \ddots & \\ 0 & & & U_{\alpha_{p+r-1}\times n_0} \end{bmatrix} \left[\begin{array}{c|c|c} I_{m_1} & & 0_{m_1\times(N_2-m_1)} \\ \hline 0_{m_2\times\alpha_2} & I_{m_2} & 0_{m_2\times(N_2-m_2-\alpha_2)} \\ \hline 0_{m_3\times(\alpha_2+\alpha_3)} & I_{m_3} & 0_{m_3\times(N_2-m_3-\alpha_2-\alpha_3)} \\ \vdots & \vdots & \vdots \\ 0_{m_p\times(\alpha_1+\alpha_2+....+\alpha_{p+r-1})} & & I_{m_{p+r-1}} \end{array} \right] \tag{50}$$

In the following development, we assume that $r = 2s + 1$, so the vector $\begin{bmatrix} \tilde{X} \\ \vdots \\ \tilde{X}^{[2p+2s]} \end{bmatrix}$ can be expressed as a product of $\hat{X}_s = \begin{bmatrix} \tilde{X} \\ \vdots \\ \tilde{X}^{[p+s]} \end{bmatrix}$ (see Appendix A.6). Therefore, the Eq. (48) yields the following one:

$$\dot{V}(X) = 2\,\mathrm{vec}^T(\hat{P}AH)U_3\ \hat{X}_s^{[2]} \tag{51}$$

where

$$U_3 = \bar{U}(U_2H_2)^+, \quad U_3 \in \Re^{(\mathrm{n_o.n_p} \times \mathrm{n_s^2})} \tag{52}$$

with

$$U_2 = \begin{bmatrix} U_{\alpha_1 \times n_s} & & & 0 \\ & U_{\alpha_2 \times n_s} & & \\ & & \ddots & \\ 0 & & & U_{\alpha_{p+s} \times n_s} \end{bmatrix} \text{ and } H_2 = \left[\begin{array}{c|c|c} I_{m_{s_1}} & & 0_{m_{s_1} \times (N_2 - m_{s_1})} \\ \hline 0_{m_{s_2} \times \alpha_2} & I_{m_{s_2}} & 0_{m_{s_2} \times (N_2 - m_{s_2} - \alpha_2)} \\ \vdots & \vdots & \vdots \\ 0_{m_{s(p+s)} \times (\alpha_2 + \ldots \alpha_{p+s})} & & I_{m_{s(p+s)}} \end{array}\right]$$

The use of lemma 2 leads to:

$$\dot{V}(X) = 2\hat{X}_s^T \underset{(n_s,n_s)}{\mathrm{mat}} (U_3^T vec(\hat{P}AH))\hat{X}_s \tag{53}$$

When considering this notation $R = U_3^T$ and following the lemma 3, the Eq. (53) can be written as:

$$\dot{V}(X) = 2\hat{X}_s^T (vec^T(\hat{P}AH) \otimes I_{n_s}) \begin{bmatrix} \mathrm{mat}_{(n_s,n_s)} R_{\cdot 1} \\ \vdots \\ \mathrm{mat}_{(n_s,n_s)} R_{\cdot n_o \cdot n_p} \end{bmatrix} \hat{X}_s \tag{54}$$

After simplification it is proved (see Appendix A.7) that $vec^{\mathrm{T}}(\hat{P}AH) \otimes I_{\mathrm{n_s}}$ is equivalent to the following:

$$vec^{\mathrm{T}}(\hat{P}AH) \otimes I_{\mathrm{n_s}} = V_0(\hat{P} \otimes I_{\mathrm{ns_0}})(AH \otimes I_{\mathrm{ns_0}}) \tag{55}$$

where

$$V_0 = vec^{\mathrm{T}}(I_{\mathrm{n_0}})U_{\mathrm{n_0 \times n_0}} \otimes I_{\mathrm{n}_s} \text{ and } n_{\mathrm{s_0}} = n_0.n_{\mathrm{s}} \tag{56}$$

Consequently, (54) and (55) allow this formulation and derive the important result:

$$\dot{V}(X) = \hat{X}_s^T (V_0(\hat{P} \otimes I_{ns_0})(AH \otimes I_{ns_0})\Im + \Im^T (AH \otimes I_{ns_0})^T (\hat{P} \otimes I_{ns_0})V_0^T)\hat{X}_s \tag{57}$$

with

$$\Im = \begin{bmatrix} \mathrm{mat}_{(n_s,n_s)} R_{\cdot 1} \\ \vdots \\ \mathrm{mat}_{(n_s,n_s)} R_{\cdot n_o \cdot n_p} \end{bmatrix}$$

A sufficient condition of the global asymptotic stability of the equilibrium ($X = 0$) is that $\dot{V}(X)$ be negative definite. This condition can be ensured if there exists a positive definite matrix Q *verifying* the result below:

Theorem 1. *Consider the nonlinear system defined by the eq.* (16) *where the integer r is odd: r = 2s + 1. If there exists a positive definite* ($n_s \times n_s$) matrix $\boldsymbol{Q}$ verifying:

$$V_0(\hat{\boldsymbol{P}} \otimes I_{\mathrm{ns_0}})(AH \otimes I_{\mathrm{ns_0}})\Im + \Im^{\mathrm{T}}(AH \otimes I_{\mathrm{ns_0}})^{\mathrm{T}}(\hat{\boldsymbol{P}} \otimes I_{\mathrm{ns_0}})V_0^{\mathrm{T}} + \mathbf{Q} < 0 \tag{58}$$

where $V_0 = vec^T(I_{n_0})U_{n_0 \times n_0} \otimes I_{n_s}$ and $n_{s_0} = n_0 \cdot n_s$
then the equilibrium of the considered system (16) *is globally asymptotically stable.*

The testing of $\hat{P} > 0$ and (58) can be made using LMIs.

Lemma 5. *The nonlinear system (16) is stable if there exists a feasible solution to LMI*:

$$\exists \hat{\boldsymbol{P}} = \hat{\boldsymbol{P}}^T, \quad \hat{\boldsymbol{P}} > 0, \quad \boldsymbol{Q} > 0, \quad V_0(\hat{\boldsymbol{P}} \otimes I_{ns_0})(AH \otimes I_{ns_0})\Im + \Im^T(AH \otimes I_{ns_0})^T(\hat{\boldsymbol{P}} \otimes I_{ns_0})V_0^T + \boldsymbol{Q} < 0 \quad (59)$$

Moreover, the Lyapunov function that demonstrates stability is given by: $V(X) = \hat{X}^T \hat{\boldsymbol{P}} \hat{X}$

6. Conclusion

An approach for global asymptotic stability of nonlinear systems with polynomial vector fields has been presented in this paper. This approach is based on polynomial Lyapunov functions which can be constructed of higher order. Sufficient conditions for the existence of such Lyapunov functions ensuring the stability of the nonlinear studied systems are proved and derived after considerable developments based on a non-redundantstate formulation, the use of the notations of the Kronecker product and the properties of the tensoriel algebra. The conditions involving PLFs can also be given as LMI feasibility tests and provide less conservative results than those generated via quadratic Lyapunov functions. For further study, these original results can be exploited for stabilisation and enlarging the attraction area of high order nonlinear systems.

Appendix A

A.1. Proof of lemma1

For $p = 2$, we have:

$$\frac{\mathrm{d}}{\mathrm{d}t}(X^{[2]}) = \frac{\mathrm{d}}{\mathrm{d}t}(X) \otimes X + X \otimes \frac{\mathrm{d}}{\mathrm{d}t}(X) = \frac{\mathrm{d}}{\mathrm{d}t}(X) \otimes X + U_{n\times n}\frac{\mathrm{d}}{\mathrm{d}t}(X) \otimes X = (I_{n^2} + U_{n\times n})(\frac{\mathrm{d}}{\mathrm{d}t}(X) \otimes X$$

$$= (I_{n^2} + U_{n\times n})\left(\sum_{i=1}^{r} F_i X^{[i]} \otimes X\right) = (I_{n^2} + U_{n\times n})\left(\sum_{i=1}^{r} F_i X^{[i+1]}\right) = \sum_{i=1}^{r} F_{i[2]} X^{[i+1]} \quad (A.1)$$

where

$$F_{i[2]} = (F_i \otimes I_n) + U_{n\times n}(F_i \otimes I_n)$$

by recurrence and when considering the relation (17) at the order p, let's demonstrate this result at the order $(p+1)$;

$$\frac{\mathrm{d}}{\mathrm{d}t}(X^{[p+1]}) = \frac{\mathrm{d}}{\mathrm{d}t}(X^{[p]}) \otimes X + X^{[P]} \otimes \frac{\mathrm{d}}{\mathrm{d}t}(X) = \frac{\mathrm{d}}{\mathrm{d}t}(X^{[p]}) \otimes X + U_{n^p\times n}\frac{\mathrm{d}}{\mathrm{d}t}(X) \otimes X^{[p]}$$

$$= \sum_{i=1}^{r} F_{i[p]} X^{[i+p-1]} \otimes X + U_{n^p\times n}\left(\sum_{i=1}^{r} F_i X^{[i]}\right) \otimes X^{[p]}$$

$$= \sum_{i=1}^{r} (F_{i[p]} \otimes I_n) X^{[i+p]} + U_{n^p\times n} \sum_{i=1}^{r} (F_i \otimes I_{n^p}) X^{[i+p]}$$

$$= \sum_{i=1}^{r} ((F_{i[p]} \otimes I_n) + U_{n^p\times n}(F_i \otimes I_{n^p})) X^{[i+p]} = \sum_{i=1}^{r} F_{i[p+1]} X^{[i+p]} \quad (A.2)$$

We have then demonstrated that $F_{i[p+1]}$ is given by the following expression as it is introduced in the Lemma 1:

$$F_{i[p+1]} = \left(F_{i[p]} \otimes I_n\right) + U_{n^p\times n}(F_i \otimes I_{n^p})$$

A.2. Proof of lemma 2

The proof of this lemma is obvious by the use of the rules of the *v*ec-function,

$$Y^T \underset{(m,n)}{\text{mat}} X = vec(Y^T \underset{(m,n)}{\text{mat}}(Z)(X) = (X^T \otimes Y^T)vec(\underset{(m,n)}{\text{mat}}(Z)) \tag{A.3}$$

Using the relation (6) it comes out:

$$Y^T \underset{(m,n)}{\text{mat}}(Z)X = (X^T \otimes Y^T)Z = (X \otimes Y^T)Z = Z^T(X \otimes Y) \tag{A.4}$$

A.3. Proof of lemma 3

with

$$Z = [z_{\cdot 1}, z_{\cdot 2} \ldots z_{\cdot q}]^T \quad \text{and} \quad A = [A_{\cdot 1}|A_{\cdot 2}|\cdots|A_{\cdot q}] \quad \text{mat}_{(n,m)}(A, Z) = \text{mat}_{(n,m)}\left(\sum_{i=1}^{q} z_i A_{\cdot i}\right)$$

$$= \sum_{i=1}^{q} z_i \text{mat}_{(n,m)}(A_{\cdot i}) = [z_1 I_n \cdots z_i I_n \cdots z_q I_n] \begin{bmatrix} \text{mat}_{(n,m)}(A_{\cdot i}) \\ \vdots \\ \text{mat}_{(n,m)}(A_{\cdot q}) \end{bmatrix} = (Z^T \otimes I_n) \begin{bmatrix} mat_{(n,m)}(A_{\cdot i}) \\ \vdots \\ mat_{(n,m)}(A_{\cdot q}) \end{bmatrix} \tag{A.5}$$

$$\text{mat}_{(n,m)}(A, Z) = (Z^T \otimes I_n) \begin{bmatrix} \text{mat}_{(n,m)}(A_{\cdot i}) \\ \vdots \\ \text{mat}_{(n,m)}(A_{\cdot q}) \end{bmatrix} \tag{A.6}$$

A.4. Proof of the relation (44)

$$\dot{\hat{X}} = A.H \begin{bmatrix} \dot{\hat{X}} \\ \tilde{X}^{[2]} \\ \vdots \\ \tilde{X}^{[p]} \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_p \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \tilde{X} \\ \vdots \\ \tilde{X}^{[r]} \end{bmatrix} \\ \begin{bmatrix} \tilde{X}^{[2]} \\ \vdots \\ \tilde{X}^{[r+1]} \end{bmatrix} \\ \vdots \\ \begin{bmatrix} \tilde{X}^{[p]} \\ \vdots \\ \tilde{X}^{[p+r-1]} \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_p \end{bmatrix} \begin{bmatrix} I_{n_1} & & 0_{n_1 \times (n_p - n_1)} \\ 0_{n_2 \times \alpha_1} & I_{n_2} & 0_{n_2 \times (n_p - n_2 - \alpha_1)} \\ 0_{n_3 \times (\alpha_1 + \alpha_2)} & I_{n_3} & 0_{n_3 \times (n_p - \alpha_1 - \alpha_2 - n_3)} \\ \vdots & \vdots & \vdots \\ 0_{n_p \times (\alpha_1 + \alpha_2 + \ldots + \alpha_{p-1})} & & I_{n_p} \end{bmatrix} \begin{bmatrix} \tilde{X} \\ \tilde{X}^{[2]} \\ \vdots \\ \tilde{X}^{[p+r-1]} \end{bmatrix} \tag{A.7}$$

and then it comes out the relation (44)

$$\dot{\hat{X}} = A.H \begin{bmatrix} \tilde{X} \\ \tilde{X}^{[2]} \\ \vdots \\ \tilde{X}^{[p+r-1]} \end{bmatrix} \tag{A.8}$$

with α_i, n_i, A_i defined in (1), (42) and (45).

A.5. Proof of Eq. (49) and (50)

$$\begin{bmatrix} \tilde{X} \\ \tilde{X}^{[2]} \\ \vdots \\ \tilde{X}^{[p+r-1]} \end{bmatrix} \otimes \begin{bmatrix} \tilde{X} \\ \tilde{X}^{[2]} \\ \vdots \\ \tilde{X}^{[p]} \end{bmatrix} = \begin{bmatrix} \tilde{X} \\ \tilde{X}^{[2]} \\ \vdots \\ \tilde{X}^{[p+r-1]} \end{bmatrix} \otimes \hat{X} \tag{A.9}$$

Using the permutation matrix rules (4), one can write:

$$\tilde{X}^{[k]} \otimes \hat{X} = U_{a_k \times n_0}(\hat{X} \otimes \tilde{X}^{[k]}) \tag{A.10}$$

Thus, the relation (A.9) becomes:

$$\begin{bmatrix} \tilde{X} \\ \tilde{X}^{[2]} \\ \vdots \\ \tilde{X}^{[p+r-1]} \end{bmatrix} \otimes \hat{X} = \begin{bmatrix} \tilde{X} \otimes \hat{X} \\ \tilde{X}^{[2]} \otimes \hat{X} \\ \vdots \\ \tilde{X}^{[p+r-1]} \otimes \hat{X} \end{bmatrix}$$

$$= \begin{bmatrix} U_{n \times n_0} \hat{X} \otimes \tilde{X} \\ U_{\alpha_2 \times n_0} \hat{X} \otimes \tilde{X}^{[2]} \\ \vdots \\ U_{\alpha p+r-1} \hat{X} \otimes \tilde{X}^{[p+r-1]} \end{bmatrix}$$

$$= \begin{bmatrix} U\alpha_1 n_0 & & & 0 \\ & U_{\alpha_2 \times n_0} & & \\ & & \ddots & \\ 0 & & & U_{\alpha p+r-1 \times n_0} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \tilde{X}^{[2]} \\ \vdots \\ \tilde{X}^{[p+1]} \end{bmatrix} \\ \begin{bmatrix} \tilde{X}^{[3]} \\ \vdots \\ \tilde{X}^{[p+3]} \end{bmatrix} \\ \begin{bmatrix} \tilde{X}^{[p+r]} \\ \vdots \\ \tilde{X}^{[2p+r-1]} \end{bmatrix} \end{bmatrix} \tag{A.11}$$

$$= \begin{bmatrix} U_{n\times n_0} & & & 0 \\ & U_{\alpha_2\times n_0} & & \\ & & \ddots & \\ 0 & & & U_{\alpha_{p+r-1}\times n_0} \end{bmatrix} \left[\begin{array}{c|c|c} I_{m_1} & & 0_{m_1\times(N_2-m_1)} \\ \hline 0_{m_2\times\alpha_2} & I_{m_2} & 0_{m_2\times(N_2-m_2-\alpha_2)} \\ \hline 0_{m_3\times(\alpha_2+\alpha_3)} & I_{m_3} & 0_{m_3\times(N_2-m_3-\alpha_2-\alpha_3)} \\ \vdots & \vdots & \vdots \\ 0_{m_p\times(\alpha_1+\alpha_2+\ldots+\alpha_{p+r-1})} & & I_{m_{p+r-1}} \end{array}\right] \begin{bmatrix} \tilde{X}^{[2]} \\ \tilde{X}^{[3]} \\ \vdots \\ \tilde{X}^{[2p+r-1]} \end{bmatrix} \tag{A.12}$$

and leads to the relations (49) and (50) (notice that $n=\alpha_1$)

A.6. Proof of the relation between $\begin{bmatrix} \tilde{X} \\ \vdots \\ \tilde{X}^{[2p+2s]} \end{bmatrix}$ and $\hat{X}_s = \begin{bmatrix} \tilde{X} \\ \vdots \\ \tilde{X}^{[p+s]} \end{bmatrix}$

With analogy to the previous development, we can write

$$\begin{aligned} \begin{bmatrix} \tilde{X} \\ \tilde{X}^{[2]} \\ \vdots \\ \tilde{X}^{[p+s]} \end{bmatrix} \otimes \begin{bmatrix} \tilde{X} \\ \tilde{X}^{[2]} \\ \vdots \\ \tilde{X}^{[p+s]} \end{bmatrix} &= \begin{bmatrix} U_{\alpha_1\times n_s} & & & 0 \\ & U_{\alpha_2\times n_s} & & \\ & & \ddots & \\ 0 & & & U_{\alpha_{p+s}\times n_s} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \tilde{X}^{[2]} \\ \vdots \\ \tilde{X}^{[p+s+1]} \end{bmatrix} \\ \begin{bmatrix} \tilde{X}^{[3]} \\ \vdots \\ \tilde{X}^{[p+s+2]} \end{bmatrix} \\ \vdots \\ \begin{bmatrix} \tilde{X}^{[p+s+1]} \\ \vdots \\ \tilde{X}^{[2p+2s]} \end{bmatrix} \end{bmatrix} \\ &= U_2 \left[\begin{array}{c|c|c} I_{m_{s_1}} & & 0_{m_{s_1}\times(N_2-m_{s_1})} \\ \hline 0_{m_{s_2}\times\alpha_2} & I_{m_{s_2}} & 0_{m_{s_2}\times(N_2-m_{s_2}-\alpha_2)} \\ \vdots & \vdots & \vdots \\ 0_{m_{s(p+s)}\times(\alpha_2+\alpha_3+\ldots\alpha_{p+s})} & & I_{m_{s(p+s)}} \end{array}\right] \begin{bmatrix} \tilde{X}^{[2]} \\ \tilde{X}^{[3]} \\ \vdots \\ \tilde{X}^{[2p+2s]} \end{bmatrix} \\ &= U_2.H_2 \begin{bmatrix} \tilde{X}^{[2]} \\ \tilde{X}^{[3]} \\ \vdots \\ \tilde{X}^{[2p+2s]} \end{bmatrix} \end{aligned} \tag{A.13}$$

Using the notations (3) we can write:

$$\begin{bmatrix} \tilde{X}^{[2]} \\ \tilde{X}^{[3]} \\ \vdots \\ \tilde{X}^{[2p+2s]} \end{bmatrix} = (U_2.H_2)^+ \begin{bmatrix} \tilde{X} \\ \tilde{X}^{[2]} \\ \vdots \\ \tilde{X}^{[p+s]} \end{bmatrix} \otimes \begin{bmatrix} \tilde{X} \\ \tilde{X}^{[2]} \\ \vdots \\ \tilde{X}^{[p+s]} \end{bmatrix} \tag{A.14}$$

A.7. Proof of relation (55)

$$\begin{aligned} vec^T(\hat{P}AH) \otimes I_{ns} &= ((AH)^T \otimes \hat{P})vec(I_{n_0}))^T \otimes I_{ns} = \left[vec^T(I_{n_0})(AH \otimes \hat{P})\right] \otimes I_{ns} \\ &= \left[vec^T(I_{n_0})(I_{n_0} \otimes \hat{P})(AH \otimes I_{n_0})\right] \otimes I_{ns} \end{aligned}$$

$$= \left[vec^T(I_{n_0})U_{n_0\times n_0}(\hat{P}\otimes I_{n_0})(AH\otimes I_{n_0})\right]\otimes I_{ns}$$
$$= V_0(\hat{P}\otimes I_{n_0}\otimes I_{ns})(AH\otimes I_{n_0}\otimes I_{ns}) = V_0(\hat{P}\otimes I_{ns_0})(AH\otimes I_{ns_0}) \quad (A.15)$$

where $V_0 = vec^T(I_{n_0})U_{n_0\times n_0}\otimes I_{n_s}$

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